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Renormalisation group study of the random Ising model in a transverse field in one dimension

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Abstract. The random Ising model in a transverse field in one dimension with Hamiltonian $H = -\sum (\Gamma_i S_i^z + J_i S_i^x S_{i+1}^x)$ is studied at $T = 0$ from a real-space renormalisation group block method which preserves duality transformations. The ground state magnetisation and the ground state energy are determined for random distributions $P(J)$ ($P(\Gamma) = N_J J^{N_J-1} / J_0^{N_J}$ for $0 < J < J_0$ and $P(J) = 0$ for $J > J_0$). A new critical behaviour corresponding to a new fixed point takes place in the presence of disorder. The crossover exponent describing the departure from the pure system behaviour is calculated. The second derivative of the ground state energy $\partial^2 E / \partial \Gamma^2$ which diverges logarithmically for the pure system is rounded in the presence of disorder but a sharp transition field still exists where the magnetisation goes to zero with an exponent β about twice as large as the pure system exponent. Comparison is made with the analytical results of McCoy and Wu for the classical equivalent random-stripped Ising 2D model.

1. Introduction

It is important to understand the effect of disorder on second-order phase transitions. For annealed systems a simple renormalisation of the critical exponents is generally expected (Fisher 1968). For strongly quenched disorder new phenomena can arise (percolation, localisation, spin glass) (Lubensky 1978). We shall here be mainly concerned with quenched disorder. (For example, we shall be interested in the transition for a magnetic system where the exchange interactions are not all the same, but satisfy some probability distribution function.) From simple heuristic arguments (Harris 1974) confirmed by ϵ expansion renormalisation group calculations (Lubensky 1975) it has been shown that weak disorder has an effect (it changes the transition of the 'pure' system) if the response of the pure system to the disorder field at the critical temperature diverges: for a magnetic system with random exchange interactions the disorder is 'relevant' if the specific heat of the pure system diverges as $C \sim \Delta T^{-\alpha}$ with $\alpha > 0$. If $\alpha < 0$ as for the three-dimensional Heisenberg model the transition of the pure system is not affected by weak disorder. Close to the critical point the fluctuations are long-range and will not see the inhomogeneities, and a well defined transition is expected. The question which remains is: When weak disorder is relevant what happens? Is there still a sharp transition, and if so, of which kind? There is no definite

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answer at the moment. Series expansion methods have been tried, so far unsuccessfully (Rushbrooke 1971, Rapaport 1972). Monte Carlo calculations seem to show a sharp unaffected transition for the diluted 2D Ising model (Zobin 1978) and for the 3D diluted Heisenberg models (Klenin 1979) where no important effect is expected from the Harris argument ($\alpha = 0$ for the 2D Ising model and $\alpha < 0$ for the 3D Heisenberg model). Real-space renormalisation group methods have been used to study bond-diluted Ising models in two and three dimensions, and no new critical behaviour was observed (Young and Stinchcombe 1976, Southern and Young 1977, Harris and Lubensky 1974, Jayaprakash *et al* 1977) (the 3D Ising model has a divergent specific heat $\alpha \sim \frac{1}{8}$ and the disorder should be relevant). Close to four dimensions renormalisation group calculations seem to show that when $\alpha > 0$ the disorder is relevant and still leads to a sharp phase transition with new critical exponents (Lubensky 1975).

When long-range correlations of the disorder are present the effects of disorder may be more drastic. When bond disorder is introduced in a model with translational invariance along one direction the Harris condition for the relevance of disorder $\alpha = 2 - d\nu > 0$ is replaced by $2 - (d-1)\nu > 0$. Thus for the two-dimensional Ising model ($d = 2, \nu = 1$) with long-range row defects $2 - (d-1)\nu = 1$ and disorder is expected to be fairly relevant, leading to a new phase transition behaviour. Such a model has already been considered by McCoy and Wu. They consider a 2D Ising model on a square lattice in which all horizontal bonds have a fixed value J_1 ; then they require vertical bonds $J_2(j)$ between rows j and $j+1$ to be equal but allow $J_2(j)$ to be a random function of j with width δJ_2 . They consider the probability distribution $P(J_2) = NJ_2^{N-1}/J_{20}^N$ for $0 < J_2 < J_{20}$ and $P = 0$ for $J_2 > J_{20}$. In a series of papers they study the free energy and the specific heat (McCoy and Wu 1968), the correlation functions (McCoy and Wu 1969, McCoy 1969) and try to extend some of their results to more general probability distributions (McCoy 1970). The main result to emerge from their analytical calculations is that the system has a well-defined critical temperature. However the specific heat does not show any sharp singularity (only an essential singularity appears in the free energy). The bulk magnetic properties could not be calculated correctly; only those at the boundary are obtained. The spontaneous boundary magnetisation disappears at the critical point with an exponent $\beta = 1$, and the boundary correlation functions show a logarithmic dependence on distance at the critical point. Up to now no other calculations (Monte Carlo or renormalisation group methods) have reproduced the results obtained by Wu and McCoy, and this new transition is not very well understood. In this paper we study by a real-space renormalisation group method the $d-1$ quantum analogue of this random classical $d=2$ Ising model. By going to some infinite anisotropic limit, the transfer matrix of the 2D statistical mechanical problem is transformed into a quantum mechanical $d=1$ Hamiltonian (Fradkin and Susskind 1978): the one-dimensional random Ising model in a transverse field

$$H = -\sum_i (\Gamma_i S_i^z + J_{i,i+1} S_i^x S_{i+1}^x), \quad (1)$$

where S_i^x and S_i^z are Pauli spin matrices. This random 1D quantum spin model has been studied exactly (Pfeuty 1979) only for the location of the transition. Another closely related quantum model, the XY spin- $\frac{1}{2}$ model in a Z field with random coupling constants, has been studied by Smith (1970), but no rigorous analysis of the new behaviour has yet been undertaken. We should mention related work in which real-space renormalisation group methods have been used for random quantum systems: the study of the localisation transition for electrons in a random potential in

two dimensions (Lee 1979, Domany and Sarker 1979) and the study of an antiferromagnetic spin- $\frac{1}{2}$ Heisenberg chain with random exchange coupling constants (Ma and Dasgupta 1979).

The real-space renormalisation group technique used in this paper is a block method introduced by Pearson (unpublished) and developed first by SLAC field theorists (Drell *et al* 1977). This method has been tested with the 1D Ising model in a transverse field (Jullien *et al* 1978) for which an exact solution exists (Pfeuty 1970). A preliminary study of the random Ising model in a transverse field (equation (1)) has been presented (Uzelac *et al* 1979). In this Letter we only calculated the phase diagram for a probability distribution of the J_i , $P(J_i)$ or of the Γ_i , $P(\Gamma_i)$ constructed from two delta functions $P(\Gamma_i) = p\delta(\Gamma_i - \Gamma_1) + (1-p)\delta(\Gamma_i - \Gamma_2)$. We now mainly study the model (1) with both random Γ_i and J_i which satisfy the probability distribution $P(J)$ ($P(\Gamma)$), $P(J) = N_J J^{N_J-1} / J_0^{N_J}$ for $0 < J < J_0$ and $P(J) = 0$ for $J > J_0$.

We use a renormalisation block method (Fernandez-Pacheco 1979) preserving duality transformations of the model with blocks of $n_s = 3$ sites. The instability of the 'pure' fixed point with respect to disorder is studied. A new fixed point is obtained which describes the new transition of the random system. This new transition is sharp in the sense that the magnetisation goes to zero for well defined values of the parameters (T , randomness) with an exponent β slightly higher than the pure system exponent ($\beta \sim 0.3$ instead of 0.125). We find a rounded peak for the second derivative of the ground state energy with respect to Γ (the logarithmic divergence of the pure system disappears). But although some marginalism is present at the new fixed point, the essential singularity which is expected in the energy, by analogy with the results of McCoy and Wu for the free energy of the 2D classical system, is not recovered. This work is divided into five parts. In § 2 we present the renormalisation group method, which preserves duality. In § 3 we give the results of this method for the non-random case; in § 4 we study the instability of the pure system fixed point with respect to disorder and calculate the crossover exponent, in § 5 we present the renormalisation group treatment of the random model: this includes the recursion relations, the fixed points and the phase diagram, the critical behaviour and the results for the ground state energy and the ground state magnetisation. In § 6 the results are discussed.

2. The renormalisation group method preserving duality

We apply to the Hamiltonian (1) a real-space renormalisation group procedure analogous to those already developed to treat uniform spin systems (Drell *et al* 1977, Jullien *et al* 1978) and in particular to treat the uniform Ising model in a transverse field in 1D. The method has been modified to preserve the duality properties of (1), that is, the symmetry between Γ and J parameters. This modification gives a real improvement in the uniform case and becomes particularly useful when applied to the disordered case. As previously, we remove such bonds as are necessary to divide the chain into independent adjacent blocks of n_s sites. In addition, using the trick of Fernandez-Pacheco (1979), we also remove a field on one site of the block in order to have the same number of sites and bonds inside the block. Here, for symmetry reasons, we have chosen to consider only n_s odd, $n_s = 2l + 1$, and to remove the field $\Gamma_{0,j}$ on the central site 0 of block j . Then the Hamiltonian for an isolated block j takes the form

$$\mathcal{H}_j = \mathcal{H}_j^+ + \mathcal{H}_j^- \quad (2)$$

where

$$\mathcal{H}_j^\pm = - \sum_{p=1}^{\pm l} (J_{p,j} S_{p-1,j}^x S_{p,j}^x + \Gamma_{p,j} S_{p,j}^z) \tag{3}$$

represents the Hamiltonian for one half of the block. p, j denotes the position of a site p ($p = -l, \dots, 0, \dots, +l$) within block j . (For simplification, we have dropped one of the indices for $J_{l,i+1}$ inside the block, as defined in figure 1 for $n_s = 5$ ($l = 2$)).

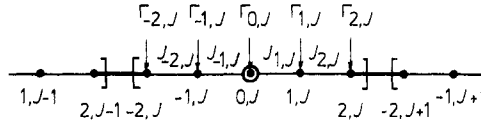


Figure 1. A block of $n_s = 5$ sites, defining the notation adopted in the text for the parameters.

Let us now consider the Hamiltonian $\mathcal{H}_{i,0}^+$ ($\mathcal{H}_{i,0}^-$) obtained from \mathcal{H}_j^+ (\mathcal{H}_j^-) by simply replacing $S_{0,j}^x$ by its upper eigenvalue $+1$

$$\mathcal{H}_{i,0}^+ = -J_{1,j} S_{1,j}^x - \Gamma_{1,j} S_{1,j}^z - \sum_{p=2}^l (J_{p,j} S_{p-1,j}^x S_{p,j}^x + \Gamma_{p,j} S_{p,j}^z). \tag{4}$$

By diagonalising $\mathcal{H}_{i,0}^+$ exactly we can determine its ground state energy E_j^+ and the expansion of the ground state wavefunction

$$\psi_j^+ = \sum_{\epsilon_p = \pm 1} \lambda_{\epsilon_1, \dots, \epsilon_l} |\epsilon_1, \dots, \epsilon_l\rangle, \tag{5}$$

where, using the S^x representation, $\epsilon_p = \pm 1$ represents the possible eigenvalues of $S_{p,j}^x$ and $|\epsilon_1, \dots, \epsilon_l\rangle$ a base in the space of dimensionality 2^l where acts $\mathcal{H}_{i,0}^+$. The energy E_j^+ and the coordinates $\lambda_{\epsilon_1, \dots, \epsilon_l}^+$ are functions of the ‘right’ parameters $J_{1,j}, \Gamma_{1,j}, \dots, J_{l,j}, \Gamma_{l,j}$. By then diagonalising $\mathcal{H}_{i,0}^-$ we find that the energy E_j^- and the coordinates $\lambda_{\epsilon_{-1}, \dots, \epsilon_{-l}}^-$ of its ground state are the same functions of the ‘left’ parameters $J_{-1,j}, \Gamma_{-1,j}, \dots, J_{-l,j}, \Gamma_{-l,j}$. Then it is straightforward to see that the ground state of the whole block is a doublet at energy $E_j = E_j^+ + E_j^-$. Two components of the doublet, denoted $|+\rangle_j$ and $|-\rangle_j$, are

$$\begin{aligned} |+\rangle_j &= \sum \lambda_{\epsilon_1, \dots, \epsilon_l}^+ \lambda_{\epsilon_{-1}, \dots, \epsilon_{-l}}^- |\epsilon_{-l}, \dots, \epsilon_{-1}, +, \epsilon_1, \dots, \epsilon_l\rangle \\ |-\rangle_j &= \sum \lambda_{\epsilon_1, \dots, \epsilon_l}^+ \lambda_{\epsilon_{-1}, \dots, \epsilon_{-l}}^- |-\epsilon_{-l}, \dots, -\epsilon_{-1}, -, -\epsilon_1, \dots, -\epsilon_l\rangle. \end{aligned} \tag{6}$$

Thus it is sufficient to diagonalise each half Hamiltonian $\mathcal{H}_{i,0}^+$ and $\mathcal{H}_{i,0}^-$ to determine completely the ground state doublet of the block.

The approximation of the method consists in retaining only the two lowest states of the block (here the doublet (6)) as a new base for the block in order to rewrite the original bonds and fields which have been dropped initially. To do so, we define a new spin S_j^i for the block, the eigenstates of S_j^x being respectively $|+\rangle_j$ and $|-\rangle_j$. By rewriting the term $-\Gamma_{0,j} S_{0,j}^z$ into $-\Gamma_j' S_j^z$, the new field Γ_j' is a function of the parameters of the block

$$\Gamma_j' = \Gamma_{0,j} \xi_\Gamma(J_{1,j}, \Gamma_{1,j}, \dots, J_{l,j}, \Gamma_{l,j}) \xi_\Gamma(J_{-1,j}, \Gamma_{-1,j}, \dots, J_{-l,j}, \Gamma_{-l,j}). \tag{7}$$

The function ξ_Γ can be determined through the coordinates

$$\xi_\Gamma(J_{1,j}, \Gamma_{1,j}, \dots, J_{l,j}, \Gamma_{l,j}) = \sum \lambda_{\epsilon_1, \dots, \epsilon_l}^+ \lambda_{-\epsilon_1, \dots, -\epsilon_l}^+ \tag{8}$$

Also, by rewriting the old spin operator $S_{p,j}^x$ we obtain the spin recursion relation

$$S_{\pm p,j}^x = \xi_{\pm p,j}^x S_j^{x'} \tag{9}$$

with

$$\xi_{0,j}^x = 1, \quad \xi_{\pm p,j}^x = \sum \epsilon_{\pm p} (\lambda_{\epsilon_{\pm 1}, \dots, \epsilon_{\pm l}}^{\pm})^2. \tag{10}$$

Using this recursion relation for the spins at the edges of two adjacent blocks we can rewrite the original interblock bond $-J_{l,j-l,j+1} S_{l,j}^x S_{-l,j+1}^x$ which transforms into $-J'_{l,j+1} S_j^{x'} S_{j+1}^{x'}$ where the new interaction $J'_{l,j+1}$ is a function of the parameters of the block

$$J'_{l,j+1} = J_{l,j;-l,j+1} \xi_J(J_{1,j}, \Gamma_{1,j}, \dots, J_{l,j}, \Gamma_{l,j}) \xi_J(J_{-1,j+1}, \Gamma_{-1,j+1}, \dots, J_{-l,j+1}, \Gamma_{-l,j+1}). \tag{11}$$

The function ξ_J is determined by

$$\xi_J(J_{1,j}, \Gamma_{1,j}, \dots, J_{l,j}, \Gamma_{l,j}) = \xi_{l,j}^x = \sum \epsilon_l (\lambda_{\epsilon_1, \dots, \epsilon_l}^{\pm})^2. \tag{12}$$

Thus, after one iteration step, the Hamiltonian takes the same form as (1) (after dropping the term E_j) but dealing with blocks and with new parameters $J'_{l,j+1}$ and Γ'_j given as a function of the old parameters by recursion relations (7) and (11). The term E_j is needed to calculate the ground state energy per site. It is also useful to consider the averaged operator

$$\sigma_j^x = \frac{1}{n_s} \sum_{p=-l}^{+l} S_{p,j}^x \tag{13}$$

which follows the recursion relation

$$\sigma_j^x = \xi_m \sigma_j^{x'}, \tag{14}$$

where

$$\xi_m = \left[1 + \sum_{p=1}^l (\xi_{p,j}^x + \xi_{-p,j}^x) \right] / n_s. \tag{15}$$

This operator will be used to calculate the x component of the magnetisation.

Let us emphasise the duality between the two recursion relations (7) and (11). Our procedure is invariant under the transformation which changes bonds into sites ($\Gamma \leftrightarrow J$). This must be reflected in the form of the expressions (7) and (11); in particular, the following identity holds:

$$\xi_\Gamma(J_1, \Gamma_1, \dots, J_l, \Gamma_l) \equiv \xi_J(\Gamma_l, J_l, \dots, \Gamma_1, J_1). \tag{16}$$

These duality properties also appear clearly when comparing the procedure applied here to the Ising model in a transverse field with the same procedure applied to the dimerised spin- $\frac{1}{2}$ XY model in 1D (Fields 1979). It has been shown that the dimerised XY model with alternate constants J_{2p} and K_{2p+1} is equivalent to two double-spaced independent and orthogonal Ising chains in a transverse field, the constants J_{2p} and K_{2p+1} being replaced by J_p and Γ_p (Jullien and Fields 1978). Removing alternately sites and bonds in the transverse field Ising model corresponds to the alternate removal of bonds J and K in the XY model. The procedure is then completely symmetric in J and K . In the treatment of the dimerised uniform XY chain blocks of odd number of sites have previously been considered for these symmetry reasons (Fields 1979).

Let us give a simple example ($n_s = 3$) where the recursion relations can be written analytically. This example will be used extensively when studying disorder. For $n_s = 3$,

ξ_Γ and ξ_J are functions of Γ_1 and J_1 only, and are given by

$$\xi_\Gamma = \Gamma_1 / (\Gamma_1^2 + J_1^2)^{1/2}; \quad \xi_J = J_1 / (\Gamma_1^2 + J_1^2)^{1/2}. \quad (17)$$

Thus the recursion relation for Γ takes the form (the index j has been dropped)

$$\Gamma' = \Gamma_0 [\Gamma_1 / (\Gamma_1^2 + J_1^2)^{1/2}] [\Gamma_{-1} / (\Gamma_{-1}^2 + J_{-1}^2)^{1/2}] \quad (18)$$

and the similar dual expression for J' . Also in this case the ground state energy of the block takes the form

$$E = -(\Gamma_1^2 + J_1^2)^{1/2} - (\Gamma_{-1}^2 + J_{-1}^2)^{1/2} \quad (19)$$

and the constant ξ_m (appearing in (14))

$$\xi_m = \frac{1}{3} \{1 + [J_1 / (J_1^2 + \Gamma_1^2)^{1/2}] + [J_{-1} / (J_{-1}^2 + \Gamma_{-1}^2)^{1/2}]\}. \quad (20)$$

For larger values of n_s , these different expressions can only be obtained by numerical computation.

Before discussing more precisely the treatment of disorder let us present briefly the main results of the present method when it is applied to the ordered case where a remarkable improvement is obtained as compared with the previous study (Jullien *et al* 1978): not only is the critical field $(\Gamma/J)_c = 1$ recovered exactly (due to duality) but also the critical exponent ν is exactly equal to one, and other exponents are improved.

3. Results in the ordered case

In this case, at each step of the renormalisation group procedure the parameters are site-independent: $J_{i,i+1} \equiv J$, $\Gamma_i \equiv \Gamma$. Then ξ_J and ξ_Γ are functions of only the dimensionless parameter Γ/J , and the duality imposes $\xi_J(\Gamma/J) \equiv \xi_\Gamma(J/\Gamma)$. The recursion relations reduce to

$$\begin{aligned} J' &= J(\xi_J(\Gamma/J))^2 \\ \Gamma' &= \Gamma(\xi_J(J/\Gamma))^2 \end{aligned} \quad (21)$$

and for the dimensionless parameter Γ/J

$$(\Gamma/J)' = (\Gamma/J)(\xi_J(J/\Gamma)/\xi_J(\Gamma/J))^2. \quad (22)$$

We always obtain an unstable fixed point $(\Gamma/J)_c = 1$ separating two trivial stable fixed points $\Gamma/J = 0$ and $\Gamma/J = +\infty$. As expected from duality arguments, the exact location of the transition $(\Gamma/J)_c = 1$ is recovered exactly. For example, in the simple case $n_s = 3$ we have

$$J' = J^3 / (J^2 + \Gamma^2), \quad \Gamma' = \Gamma^3 / (J^2 + \Gamma^2) \quad (23)$$

and

$$\Gamma'/J' = (\Gamma/J)^3. \quad (24)$$

The exponent ν which describes the divergence of the coherence length at the transition is obtained by linearising the recursion relation (22) near $\Gamma/J = 1$. It can be seen from formula (24) that for $n_s = 3$ we recover the exact result $\nu = 1$. This result, independent of n_s , has been verified by numerical computation up to $n_s = 17$.

The 'dynamical' exponent z which gives the dilatation of energy at the transition is obtained from the renormalisation of J (or Γ) for $\Gamma/J = 1$:

$$z = -2 \ln \xi_J(\Gamma/J = 1) / \ln n_s. \tag{25}$$

For $n_s = 3$ we find

$$z = \ln 2 / \ln 3 = 0.6309 \dots \tag{26}$$

Using the equivalence between the Ising model in a transverse field at the transition and the uniform XY chain (24) (Jullien and Fields 1978) where the renormalisation group procedure can be performed analytically, we are able to derive an analytical formula for z :

$$z = \ln((n_s + 1)/2) / \ln n_s. \tag{27}$$

This formula will be derived elsewhere (Pfeuty, Jullien and Penson, to be published).

Expression (27) gives an idea of the very slow convergence of z towards the exact value $z = 1$ when $n_s \rightarrow \infty$. As generally observed with this kind of method, when they tend to their exact value, the exponents converge as $1/\ln n_s$. The plot of z as a function of $1/\ln n_s$ is given in figure 2.

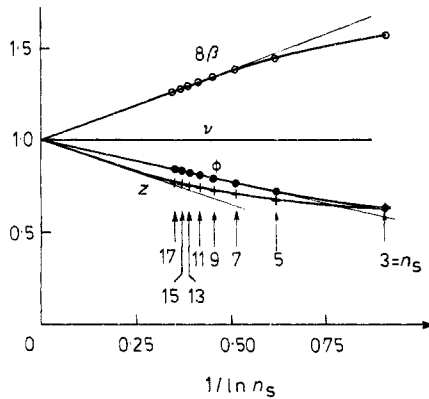


Figure 2. The exponents β , z and ν of the ordered system and the crossover exponent ϕ plotted as a function of $1/\ln n_s$ up to $n_s = 17$.

The x component of the magnetisation can be obtained numerically by integrating formula (14) up to the fixed point. We obtain

$$\begin{aligned} \langle \sigma^x \rangle &= \lim_{n \rightarrow \infty} \xi_m^{(0)} \dots \xi_m^{(n)} && \text{for } \Gamma/J < (\Gamma/J)_c \\ \langle \sigma^x \rangle &= 0 && \text{for } \Gamma/J > (\Gamma/J)_c. \end{aligned} \tag{28}$$

The exponent β can be extracted from the numerical computation of $\langle \sigma^x \rangle$ near $(\Gamma/J)_c$. The curve giving $\langle \sigma^x \rangle$ as a function of Γ/J is given in figure 5 for $n_s = 3$ (dotted curve).

The exponent η which describes the power law decay of the x - x correlation function at the transition in the expression $\langle S_i^x S_{i+R}^x \rangle \sim R^{-\eta}$ is obtained from the renormalisation of σ^x at the transition and is given by

$$\eta = -2 \ln(\xi_m(\Gamma/J = 1)) / \ln n_s. \tag{29}$$

We have verified, for all n_s , the scaling law $2\beta = \eta\nu$ which reduces here to $2\beta = \eta$, and we have preferred to calculate β directly from η (this gives a more precise result). For $n_s = 3$ we find

$$\beta = \nu/2 = -\ln((1 + \sqrt{2})/3)/\ln 3 = 0.1977 \dots \quad (30)$$

The results for β obtained by numerical computation up to $n_s = 17$ are reported as a function of $1/\ln n_s$ in figure 2. We observe a good linearity with the exact value $\beta = \frac{1}{8}$ obtained for $n_s \rightarrow \infty$.

We can also calculate the ground state energy per site by cumulating the constant $E_j^{(n)}$ divided by the number of sites at step n :

$$-\frac{E}{N} = \lim_{n \rightarrow \infty} \sum \frac{-E_j^{(n)}}{n_s^n}. \quad (31)$$

In the case $n_s = 3$ this formula gives

$$-\frac{E}{N} = 2 \lim_{n \rightarrow \infty} \sum \frac{(\Gamma^{(n)2} + J^{(n)2})^{1/2}}{3^n}. \quad (32)$$

The second derivative of the ground state energy with respect to the transverse field, which corresponds to the specific heat in the classical analogue, has been plotted for $n_s = 3$ in figure 6 (dotted curve). We observe a much more marked peak than with the previous approach (Jullien *et al* 1978). The reason is that the α exponent giving the divergence is here found to be positive. From scaling arguments α is given by $\alpha = 2 - (1 + z)\nu = 1 - z$. Here for $n_s = 3$ we find $\alpha = 0.36$, which corresponds to a strong singularity.

The different numerical results in the ordered case are listed in table 1 for $n_s = 3$, $n_s = 5$ and for the upper n_s value for which we have performed numerical computations, $n_s = 17$.

4. Generalities on disorder and crossover exponent

At a given step of the iterative process described in the preceding section, the constants $J_{i,i+1}$ and Γ_i entering the Hamiltonian are supposed to be distributed with some probability distribution. As a first approximation we shall assume that all these random parameters are distributed independently from one another with probability distribution $P_J(J)$ for the $J_{i,i+1}$ and $P_\Gamma(\Gamma)$ for the Γ_i . In principle the renormalised probability distributions $P'_J(J')$ and $P'_\Gamma(\Gamma')$ could be calculated from the renormalisation equations (7) and (11). For example, for $n_s = 3$, $P'_\Gamma(\Gamma')$ is given by

$$P'_\Gamma(\Gamma') = \int P_\Gamma(\Gamma_0) d\Gamma_0 \int P_\Gamma(\Gamma_1) d\Gamma_1 \int P_\Gamma(\Gamma_{-1}) d\Gamma_{-1} \int P_J(J_1) dJ_1 \\ \times \int P_J(J_{-1}) dJ_{-1} \delta\left(\Gamma' - \Gamma_0 \frac{\Gamma_1}{(\Gamma_1^2 + J_1^2)^{1/2}} \frac{\Gamma_{-1}}{(\Gamma_{-1}^2 + J_{-1}^2)^{1/2}}\right). \quad (33)$$

In fact it is numerically too complicated to follow precisely the entire shape of the probability distributions and we must consider some simplifying truncations. We shall assume that $P_J(J)$ ($P_\Gamma(\Gamma)$) conserves its reduced shape through the iterative process, and we shall assume that this shape depends on only two parameters. These two parameters could be the position of the average J and the standard deviation $\sigma_J = (\overline{J^2} - (\overline{J})^2)^{1/2}$. We

Table 1. Location of the transition and exponents ν , z and β for the uniform Ising model in a transverse field as found with our method for $n_s = 3$ and 5 and for the larger n_s value for which we have performed numerical computation $n_s = 17$. The results are compared with the exact results. The last line gives the crossover exponent, which gives the instability of the ordered fixed point $(\Gamma/J)_c = 1$, in the presence of disorder, as defined in the text. The exact result $\phi = 1$ comes from Mc Coy and Wu (1968).

	n_s			exact $n_s \rightarrow \infty$
	3	5	17	
$(\Gamma/J)_c$	1	1	1	1
ν	1	1	1	1
$z = \frac{\ln[(n_s + 1)/2]}{\ln n_s}$	0.631	0.683	0.775	1
$\beta = \eta/2$	0.198	0.182	0.161	0.125
ϕ	0.631	0.727	0.840	1

then determine implicitly the renormalisation group recursion relations for the two parameters by fixing the two first moments of the distribution for J' and Γ' . We thus obtain the relations

$$\begin{aligned} \bar{J}' &= \bar{J}(\bar{\xi}_J)^2 & \bar{\Gamma}' &= \bar{\Gamma}(\bar{\xi}_\Gamma)^2 \\ \bar{J}'^2 &= \bar{J}^2(\bar{\xi}_J^2)^2 & \bar{\Gamma}'^2 &= \bar{\Gamma}^2(\bar{\xi}_\Gamma^2)^2, \end{aligned} \tag{34}$$

where the upper bar means a statistical average over the variables. For example

$$\bar{\xi}_J = \int P_J(J_1) dJ_1 \int P_\Gamma(\Gamma_1) d\Gamma_1 \dots \int P_J(J_l) dJ_l \int P_\Gamma(\Gamma_l) d\Gamma_l \xi_J(J_1, \Gamma_1, \dots, J_l, \Gamma_l). \tag{35}$$

Before studying a particular example of probability distribution (this will be done in the next section) we would like to derive the crossover exponent ϕ which gives the instability of the ordered unstable fixed point in the presence of disorder. This calculation is independent of the shape of the probability distribution. Let us introduce small 'reduced' standard deviations σ_J/\bar{J} and $\sigma_\Gamma/\bar{\Gamma}$ around the mean values \bar{J} and $\bar{\Gamma}$ at the unstable ordered fixed point $\bar{J} = \bar{\Gamma}$ (obtained for $\sigma_J = \sigma_\Gamma = 0$), and let us derive the renormalisation equations for these quantities. For small standard deviations, we obtain from (34)

$$\left(\frac{\sigma_J'}{\bar{J}'}\right)^2 = \left(\frac{\sigma_J}{\bar{J}}\right)^2 + 2\left(\frac{\sigma_{\xi_J}}{\bar{\xi}_J}\right)^2, \tag{36}$$

where

$$\sigma_{\xi_J}^2 = \overline{\xi_J^2} - (\bar{\xi}_J)^2. \tag{37}$$

A similar equation holds for $\sigma_\Gamma/\bar{\Gamma}$. The quantity $\sigma_{\xi_J}^2$ can be calculated from (35) and is given by

$$\sigma_{\xi_J}^2 = \sigma_J^2 \sum_{p=1}^l \left(\frac{\partial \xi_J}{\partial J_p}\right)_0^2 + \sigma_\Gamma^2 \sum_{p=1}^l \left(\frac{\partial \xi_J}{\partial \Gamma_p}\right)_0^2, \tag{38}$$

where the index zero means that the differentiations have been performed around the

point $J_1 = J_2 \dots = J_l = \Gamma_1 = \Gamma_2 = \dots = \Gamma_l$. We then obtain

$$\begin{aligned} \left(\frac{\sigma'_J}{\bar{J}}\right)^2 &= \left(1 + \frac{2}{\xi_0^2} \sum \left(\frac{\partial \xi_J}{\partial J_p}\right)_0^2\right) \left(\frac{\sigma_J}{\bar{J}}\right)^2 + \frac{2}{\xi_0^2} \sum \left(\frac{\partial \xi_J}{\partial \Gamma_p}\right)_0^2 \left(\frac{\sigma_\Gamma}{\bar{\Gamma}}\right)^2, \\ \left(\frac{\sigma'_\Gamma}{\bar{\Gamma}}\right)^2 &= \left(1 + \frac{2}{\xi_0^2} \sum \left(\frac{\partial \xi_\Gamma}{\partial \Gamma_p}\right)_0^2\right) \left(\frac{\sigma_\Gamma}{\bar{\Gamma}}\right)^2 + \frac{2}{\xi_0^2} \sum \left(\frac{\partial \xi_\Gamma}{\partial J_p}\right)_0^2 \left(\frac{\sigma_J}{\bar{J}}\right)^2, \end{aligned} \tag{39}$$

where ξ_0 is the common value of ξ_J and ξ_Γ for $J_1 = J_2 \dots = J_l = \Gamma_1 = \Gamma_2 = \dots = \Gamma_l$. This coupled set of recursion relations has the same corresponding coefficients from duality properties; thus, the larger eigenvalue, which gives the renormalisation of $(\sigma_J/\bar{J})^2 + (\sigma_\Gamma/\bar{\Gamma})^2$, is given by

$$\lambda = 1 + \frac{2}{\xi_0^2} \sum \left(\left(\frac{\partial \xi_J}{\partial J_p}\right)_0^2 + \left(\frac{\partial \xi_\Gamma}{\partial \Gamma_p}\right)_0^2 \right). \tag{40}$$

We observe that $\lambda > 1$, thus the ordered fixed point is unstable against disorder. This new instability is traduced by an exponent $\ln \lambda / \ln n_s$ which can be compared with the exponent $1/\nu$ giving the instability in Γ/J . Their ratio defines the crossover exponents ϕ ; since here $\nu = 1$, we obtain

$$\phi = \ln \lambda / \ln n_s. \tag{41}$$

This crossover exponent has been numerically computed up to $n_s = 17$, and the results are reported in table 1 and figure 2. We observe a good convergence toward $\phi = 1$ for $n_s \rightarrow \infty$. The exact result $\phi = 1$ is expected from a simple argument (Harris 1974, Lubensky 1978). It gives for the random 2D Ising model $\phi = 2 - d\nu = 0$, while for the random ‘stripped’ 2D Ising model (Lubensky 1978) $\phi = 2 - (d - 1)\nu = 1$. In the case of the quantum equivalent, the random Ising model in a transverse field, one expects also $\phi = 1$.

5. Renormalisation group treatment of disorder for $n_s = 3$

5.1. Probability distribution and RG equations

We now present an extensive study of the renormalisation treatment of the Hamiltonian (1) by using blocks of three sites ($n_s = 3$) and by choosing the following shape for the probability distributions:

$$\begin{aligned} P_J(J) &= N_J (J^{N_J-1} / J_0^{N_J}) && \text{for } 0 < J < J_0 \\ P_J(J) &= 0 && \text{for } J > J_0. \end{aligned} \tag{42}$$

An analogous equation is given for Γ .

This distribution considers only positive values of the variable. This is justified by the fact that, after simple gauge transformations, the original disorder can always be reduced to a disorder in the amplitudes of the parameters. On the other hand, taking this particular shape will not reduce the generality of the results since we are only renormalising the two first moments. As a check, the same calculations have been performed with the distribution $P_J(J) = [1/(2\pi)^{1/2} \Sigma_J] \exp[-\ln^2(J/J_0)/2 \Sigma_J^2]$ leading to the same qualitative results. The distribution (42) was used by McCoy and Wu (1968) in their classical 2D model.

This shape depends on two parameters J_0 and N_J which are linked with \bar{J} and σ_J through the equations

$$\begin{aligned} \bar{J} &= J_0[N_J/(1+N_J)] \\ \bar{J}^2 &= \sigma_J^2 + (\bar{J})^2 = J_0^2[N_J/(2+N_J)]. \end{aligned} \tag{43}$$

Notice the relation between the parameter N_J and the reduced standard deviation σ_J/\bar{J} :

$$(\sigma_J/\bar{J})^2 = 1/N_J(N_J+2). \tag{44}$$

When $N_J \rightarrow \infty$, $\sigma_J/\bar{J} \rightarrow 0$, we obtain a peaked distribution at J_0 (ordered limit). When $N_J \rightarrow 0$ the probability distribution becomes peaked near $J = 0$ but with a large tail leading to $\sigma_J/\bar{J} \rightarrow \infty$. So, even if the distribution becomes strongly peaked at $J = 0$ the limit $N_J \rightarrow 0$ corresponds to strong disorder ($\sigma_J/\bar{J} \rightarrow \infty$).

By using this probability distribution the renormalisation group equation for $J_0, \Gamma_0, N_J, N_\Gamma$ can be obtained from (34):

$$\frac{N'_J J_0}{1+N'_J} = \frac{N_J J_0}{1+N_J} (\bar{\xi}_J)^2; \quad \frac{N'_J J_0^2}{2+N'_J} = \frac{N_J J_0^2}{2+N_J} (\bar{\xi}_J^2)^2 \tag{45}$$

and the dual equation for Γ .

The averaged quantities $\bar{\xi}_J$ and $\bar{\xi}_J^2$ are double integrals in the case $n_s = 3$. For example $\bar{\xi}_J$ is given by

$$\bar{\xi}_J = N_J N_\Gamma \int_0^{J_0} \int_0^{\Gamma_0} \frac{J}{(J^2 + \Gamma^2)^{1/2}} \frac{J^{N_J-1}}{J_0^{N_J}} \frac{\Gamma^{N_\Gamma-1}}{\Gamma_0^{N_\Gamma}} dJ d\Gamma. \tag{46}$$

By standard transformations the double integrals can be transformed into simple integrals, leading to

$$\begin{aligned} \bar{\xi}_J &= \frac{N_J N_\Gamma}{N_J + N_\Gamma} \int_0^1 \left(\frac{J_0 u}{(\Gamma_0^2 + J_0^2 u^2)^{1/2}} u^{N_J-1} + \frac{J_0}{(\Gamma_0^2 u^2 + J_0^2)^{1/2}} u^{N_\Gamma-1} \right) du \\ \bar{\xi}_J^2 &= \frac{N_J N_\Gamma}{N_J + N_\Gamma} \int_0^1 \left(\frac{J_0^2 u^2}{\Gamma_0^2 + J_0^2 u^2} u^{N_J-1} + \frac{J_0^2}{\Gamma_0^2 u^2 + J_0^2} u^{N_\Gamma-1} \right) du. \end{aligned} \tag{47}$$

These integrals have been used in the numerical computations.

5.2. Fixed points and phase diagram

The parameter space contains the three independent parameters $\Gamma_0/J_0, N_\Gamma$ and N_J . In this space we find a critical surface $(\Gamma_0/J_0)_c = f(N_\Gamma, N_J)$ which determines the phase diagram of the disordered system. This surface has been represented in figure 3 by using the set of coordinates $\Gamma_0/J_0, 1/N_\Gamma, 1/N_J$. In the figure the Γ_0/J_0 axis which corresponds to $1/N_\Gamma = 1/N_J = 0$ represents the ordered system and it contains the ordered fixed point A_0 ($\Gamma_0/J_0 = 1, 1/N_\Gamma = 0, 1/N_J = 0$). In the presence of disorder this whole axis becomes unstable. By studying numerically the renormalisation group recursion relations we find that if we start with $\Gamma_0/J_0 < (\Gamma_0/J_0)_c$ (and not on the ordered axis) we end up with $\Gamma_0/J_0 \rightarrow 0, N_\Gamma/N_J \rightarrow 0$, while if we start with $\Gamma_0/J_0 > (\Gamma_0/J_0)_c$, we end up with $\Gamma_0/J_0 \rightarrow \infty, N_\Gamma/N_J \rightarrow \infty$. If we start now just on the surface, we follow a trajectory entirely contained in the surface which ends up at a new 'disordered' fixed point A_D ($\Gamma_0/J_0 = 1, N_\Gamma = 0, N_J = 0$) which is thus stable on the surface but unstable

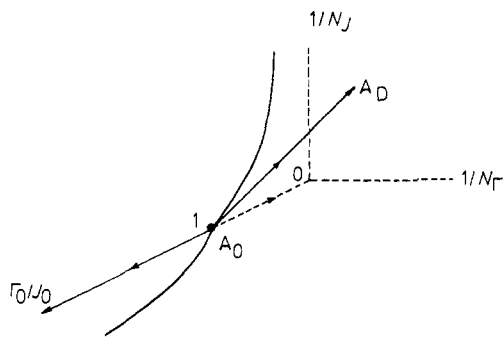


Figure 3. The critical surface in the parameter space $\{\Gamma_0/J_0, 1/N_J, 1/N_\Gamma\}$ obtained with blocks of three sites. The sections by the planes $1/N_J = 0$ and $1/N_\Gamma = 0$ are represented by the bold curves. A_0 is the ordered fixed point and A_D is the disordered fixed point. The trivial self-dual trajectory A_0A_D (contained in the critical surface) is shown.

elsewhere. The disordered fixed point is self-dual. A trivial self-dual trajectory $\Gamma_0/J_0 = 1, N_\Gamma = N_J$ going from A_0 to A_D is represented in figure 3.

It is interesting to compare the phase diagram that we have obtained here by approximation with the exact condition of Pfeuty (1979) for the annulation of the gap:

$$\pi J_{i,i+1} = \pi \Gamma_i \tag{48}$$

which yields

$$\overline{\ln J} = \overline{\ln \Gamma}. \tag{49}$$

For the probability distributions used here this condition can be written as

$$\Gamma_0/J_0 = \exp\left(-\left(\frac{1}{N_J} - \frac{1}{N_\Gamma}\right)\right). \tag{50}$$

This 'exact' phase diagram has been represented by the dashed curve of figure 4 and compared with the sections of our critical surface for constant N_Γ values $N_\Gamma = 0.1, N_\Gamma = 1$. The curve $N_\Gamma = 10$ which is almost superposed on the exact curve is not

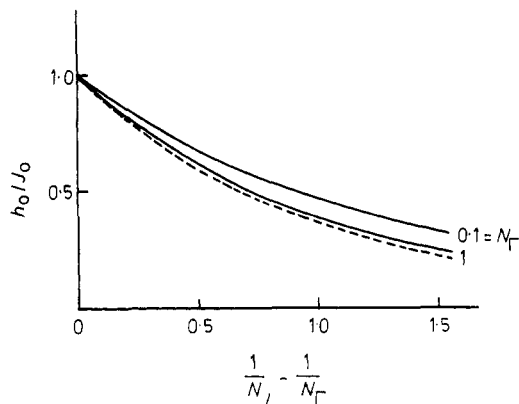


Figure 4. Section of the critical surface, for constant N_Γ values, as a plot of h_0/J_0 versus $(1/N_J) - (1/N_\Gamma)$. The exact condition of Pfeuty (1979) is represented by the dashed curve.

represented. One can see that the exact phase diagram is well reproduced for large N_Γ and N_J values. Errors become important only when we consider very strong disorder.

5.3. Linearisation near the new fixed point

As shown in the preceding paragraph, the transition in Γ_0/J_0 for the disordered system is governed by the new fixed point A_D ($\Gamma_0/J_0 = 1, N_J = N_\Gamma = 0$). One can linearise the recursion relations (45) near this fixed point by using the following expansions of the integrals (47):

$$\begin{aligned} \bar{\xi}_J &\approx \frac{N_J}{N_\Gamma + N_J} + \frac{N_J N_\Gamma}{N_\Gamma + N_J} \ln(2\Gamma_0/J_0) + O(N^2) \\ \bar{\xi}_J^2 &\approx \frac{N_J}{N_\Gamma + N_J} + \frac{N_J N_\Gamma}{N_\Gamma + N_J} \ln(\Gamma_0/J_0) + O(N^2). \end{aligned} \tag{51}$$

When adopting the following notations

$$N = (N_J + N_\Gamma)/2 \quad n = (N_J - N_\Gamma)/(N_J + N_\Gamma) \quad \Gamma_0/J_0 = 1 + \epsilon \tag{52}$$

the linearised renormalisation equations near $n = 0, N = 0, \epsilon = 0$ are decoupled in the first order:

$$n' \sim 3n; \quad N' \sim \frac{1}{4}N; \quad \epsilon' \sim \epsilon. \tag{53}$$

Thus we find that the new fixed point is stable against variations in N , unstable against the variations in n and *marginal* against the variations in Γ_0/J_0 . The larger eigenvalue corresponds to the relevant parameter n which is related to the difference between the standard deviations of J and Γ ; the corresponding exponent ν_D is equal to 1. The instability of n has already been observed in the preceding paragraph, since for $\Gamma_0/J_0 < (\Gamma_0/J_0)_c, N_\Gamma/N_J \rightarrow 0$, that is $n \rightarrow 1$, while for $\Gamma_0/J_0 > (\Gamma_0/J_0)_c, N_\Gamma/N_J \rightarrow \infty$, that is $n \rightarrow -1$. Further expansion shows that the marginality in ϵ is stable. This does not yield an infinite exponent ν as found by Mc Coy and Wu (1968) but still $\epsilon \rightarrow 0$ very very slowly near the fixed point (depending on the direction of approach to the fixed point).

At the new fixed point, one can define a dynamical exponent z_D linked with the dilatation of $\bar{\Gamma}$ and \bar{J} ; since $\bar{\Gamma}' = \bar{\Gamma}/4, \bar{J}' = \bar{J}/4$ in A_D , one obtains $z_D = \ln 4/\ln 3 = 1.26$.

5.4. Magnetisation

As in the ordered case the magnetisation is calculated by using the averaged operator $\sigma_j^x = (1/n_s) \sum S_{j,p}^x$. Only now at each step we take also the average over the disorder, so that σ^x follows the recursion relation

$$\sigma^x = \xi_m \sigma^{x'} \tag{54}$$

with, for $n_s = 3$,

$$\xi_m = (1 + 2\bar{\xi}_J)/n_s. \tag{55}$$

The magnetisation has been plotted in figure 5 as a function of Γ_0/J_0 for different values of $N_\Gamma = N_J$. For $N_\Gamma = N_J$ the location of the transition $(\Gamma_0/J_0)_c = 1$ is unchanged and always corresponds to vanishing magnetisation. For large values of $N_\Gamma = N_J$ the curve resembles the ordered one, but for small values of $N_\Gamma = N_J$ the magnetisation is reduced as would be expected in the presence of disorder. The analysis of the curves

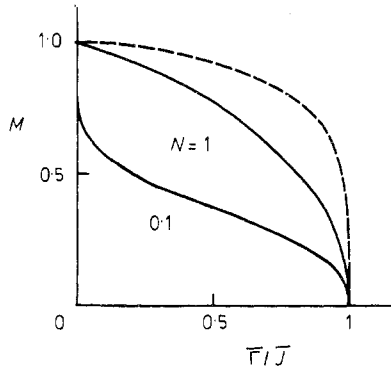


Figure 5. Result for the x magnetisation with blocks of three sites as a plot of $M = \langle S^x \rangle$ versus $\bar{\Gamma}/\bar{J}$ for different values of $N = N_\Gamma = N_J$. The dashed curve corresponds to the ordered limit ($N \rightarrow \infty$).

near $(\Gamma_0/J_0) = (\Gamma_0/J_0)_c$ gives a new exponent $\beta_D = 0.370$ different from the ordered one ($\beta = 0.198$ for $n_s = 3$).

At the transition the trajectories flow to the new fixed point $N_\Gamma = N_J \rightarrow 0$ in which $\xi_J = \frac{1}{2}$ (see equation (51)) so that at the transition the exponent d_D giving the spin dilatation is given by $d_D = -\ln(\xi_m)/\ln 3 = 0.37$. We could define as for ordered systems an exponent $\eta_D = 2d_D = 0.74$, but here, in the presence of disorder, there is no reason why this exponent should give the power-law decay of the spin correlation function.

5.5. Energy

The ground state energy per site is also calculated as in the ordered case, with the addition of averaging over disorder at each step. Thus (32) is replaced by

$$-E/N = \lim_{n \rightarrow \infty} (-\bar{E}_j^{(n)}/3^n), \quad (56)$$

where \bar{E}_j is calculated at each step of the iterative process as a double integral which can be reduced to a simple integral

$$-\bar{E}_j = 2 \frac{N_J N_\Gamma}{1 + N_J + N_\Gamma} \int_0^1 [(\Gamma_0^2 + J_0^2 u^2)^{1/2} u^{N_J-1} + (\Gamma_0^2 u^2 + J_0^2)^{1/2} u^{N_\Gamma-1}] du. \quad (57)$$

In figure 6 we have plotted the second derivative of $-E/N$ with respect to the applied field. This quantity corresponds to the specific heat in the classical analogue.

We observe that the peak of the specific heat disappears in the presence of disorder as already found by McCoy and Wu. A possible explanation of the disappearance of the peak could be obtained by estimating the exponent α_D from the scaling law $\alpha_D = 2 - \nu_D(1 + z_D) = -0.26$. α_D is here negative.

6. Discussion

The results presented above can be examined as the next step in a study of the application of a quantum renormalisation group method to a disordered system that is precisely equivalent to the McCoy and Wu model (1968). The problem has already

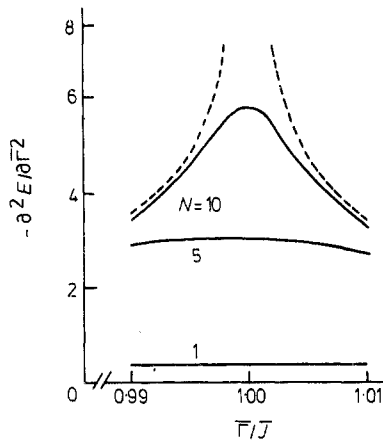


Figure 6. Result for the second derivative of the ground state energy with respect to $\bar{\Gamma}$ for different values of $N = N_{\Gamma} = N_J$. The dashed curve corresponds to the ordered limit ($N \rightarrow \infty$).

been treated by a non-dual renormalisation group method, where only the transverse field was random, distributed through two delta functions, while the Ising interaction was kept homogeneous (Uzelac *et al* 1979). Such a crude approximation was sufficient to give the correct phase diagram, but not the expected critical behaviour.

The present approximation is still very simple (probability distributions for Γ and J still decorrelated and fixed by only two moments) in order to allow, as far as possible, an analytical study. However we observe now a real improvement compared with the previous study: we obtain the right crossover exponent and some other new results which agree with expectations or exact results of McCoy and Wu (1968). In particular the second derivative of the ground state energy with respect to the transverse field is rounded at the transition as was found for the specific heat in the classical equivalent. The magnetisation, which is here calculated in the whole low- Γ phase, is reduced in the presence of disorder and the corresponding β_D exponent is found to be larger (by a factor of almost two) than in the ordered case. For the McCoy and Wu model, only an upper limit of β_D (corresponding to the 'edge' magnetisation) was found which was twice as large as the corresponding ordered exponent. However, an important discrepancy remains concerning the exponent ν_D and the new fixed point. Our simple approximation is no longer justified in the limit of very large standard deviations. Thus, instead of marginality, we obtain a set of decoupled (in the leading order) equations, where quantity relative to disorder is strongly relevant while Γ/J shows a stable marginality. Further improvements which should be carried out in the future are the introduction of correlations between Γ and J parameters, by taking into account a larger number of levels. This constitutes another kind of investigation which implies much more complex numerical computations.

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